Exercises for Chapter 2: Linear Algebra

July 13, 2016

Exercises

1. The expression $\alpha \boldsymbol{u}$ for $\alpha \in \mathbb{R}$ and unit vector $\boldsymbol{u} \in \mathbb{R}^n$ defines a line of points that may be obtained by varying the value of α. Derive an expression for the point $\boldsymbol{y}$ that lies on this line that is as close as possible to an arbitrary point $\boldsymbol{x} \in \mathbb{R}^n$. This operation of replacing a point by its nearest member within some set is called *projection*.

   Exercise contributed by Ian Goodfellow

Solutions

1. We begin by defining the distance from $\boldsymbol{y}$ to $\boldsymbol{x}$. We would like to find the $\boldsymbol{y}$ that minimizes this distance:

$$||\boldsymbol{x} - \boldsymbol{y}||^2. \tag{1}$$

   Next, we need to enforce the constraint that $\boldsymbol{y}$ lies on the line defined by $\alpha \boldsymbol{u}$. We can do this simply by defining $\boldsymbol{y}$ to be $\alpha \boldsymbol{u}$.

$$||\boldsymbol{x} - \alpha \boldsymbol{u}||^2. \tag{2}$$

   Next, we expand the expression:

$$
\begin{aligned}
&||\boldsymbol{x} - \alpha \boldsymbol{u}||^2 && (3)\\
=&(\boldsymbol{x} - \alpha \boldsymbol{u})^\top (\boldsymbol{x} - \alpha \boldsymbol{u}) && (4)\\
=&\boldsymbol{x}^\top \boldsymbol{x} - 2\alpha \boldsymbol{x}^\top \boldsymbol{u} + \alpha^2 \boldsymbol{u}^\top \boldsymbol{u} && (5)\\
=&\boldsymbol{x}^\top \boldsymbol{x} - 2\alpha \boldsymbol{x}^\top \boldsymbol{u} + \alpha^2. && (6)
\end{aligned}
$$

   In the last line, we used the fact that $\boldsymbol{u}$ is a unit vector to make the simplification $\boldsymbol{u}^\top \boldsymbol{u} = 1$.

   We can minimize this distance by taking the derivative with respect to α and setting it to zero:

$$
\begin{aligned}
&-2\boldsymbol{x}^\top \boldsymbol{u} + 2\alpha = 0 && (7)\\
\Rightarrow &\alpha = \boldsymbol{x} \top \boldsymbol{u}. && (8)
\end{aligned}
$$

Recalling that $\boldsymbol{y} = \alpha \boldsymbol{u}$, we can conclude that $\boldsymbol{y} = \boldsymbol{x}^\top \boldsymbol{u}\boldsymbol{u}$.

Solution contributed by Ian Goodfellow